Cours pour Master 1 systemes dynamiques II

Cours pour Master 1 systemes dynamiques II

Présenté par
R.BENTERKI

Université Mohamed El Bachir El Ibrahimi Bordj Bou Arréridj

March 26, 2020

- ☞ Definition
- ☞ Theorem 1 (Conservation of Energy) and its proof.
- ☞ Lemma and its proof
- ☞ Definition 2. A critical point
- ☞ Theorem 1 and its proof
- ☞ Theorem 2 and Example

Chapter I

- ☞ Definition
- ☞ Theorem 1 (Conservation of Energy) and its proof.
- ☞ Lemma and its proof
- ☞ Definition 2. A critical point
- ☞ Theorem 1 and its proof
- ☞ Theorem 2 and Example

- ☞ Definition
- ☞ Theorem 1 (Conservation of Energy) and its proof.
- ☞ Lemma and its proof
- ☞ Definition 2. A critical point
- ☞ Theorem 1 and its proof
- ☞ Theorem 2 and Example

Chapter I

- ☞ Definition
- ☞ Theorem 1 (Conservation of Energy) and its proof.
- ☞ Lemma and its proof
- ☞ Definition 2. A critical point
- ☞ Theorem 1 and its proof
- ☞ Theorem 2 and Example

Chapter I

- ☞ Definition
- ☞ Theorem 1 (Conservation of Energy) and its proof.
- ☞ Lemma and its proof
- ☞ Definition 2. A critical point
- ☞ Theorem 1 and its proof
- ☞ Theorem 2 and Example

Whats a Hamiltonian system?

Definition of
Hamiltonian system

Whats a Hamiltonian system?

Definition of
Hamiltonian system

Whats a Hamiltonian system?

Definition **of**
Hamiltonian system

Whats a Hamiltonian system?

Definition **of**
Hamiltonian system

In this section we study two interesting types of systems which arise in physical problems and from which we draw a wealth of examples of the general theory.

Definition

Let E be an open subset of $\mathbb{R}^{2n}$ and let $H \in C^2(E)$ where $H = H(x, y)$ with $x, y \in \mathbb{R}$. A system of the form

$$\dot{x} = -\frac{\partial H}{\partial y}, \qquad \dot{y} = \frac{\partial H}{\partial x}. \tag{1}$$

In this section we study two interesting types of systems which arise in physical problems and from which we draw a wealth of examples of the general theory.

Definition

Let E be an open subset of $\mathbb{R}^{2n}$ and let $H \in C^2(E)$ where $H = H(x, y)$ with $x, y \in \mathbb{R}$. A system of the form

$$\dot{x} = -\frac{\partial H}{\partial y}, \qquad \dot{y} = \frac{\partial H}{\partial x}. \qquad (1)$$

Where

$$\frac{\partial H}{\partial x} = (\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, ..., \frac{\partial H}{\partial x_n})^T$$

and

$$\frac{\partial H}{\partial y} = (\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, ..., \frac{\partial H}{\partial y_n}),$$

is called a Hamiltonian system with n degrees of freedom on E.

Example

The Hamiltonian function

$$H(x, y) = x^2 - y^2 + z^2 - w^2 + 3.$$

is the energy function for the spherical pendulum

$$\begin{aligned} \dot{x} &= 2z, \\ \dot{y} &= -2w, \\ \dot{z} &= -2x, \\ \dot{w} &= -2y. \end{aligned} \tag{2}$$

Where

$$\frac{\partial H}{\partial x} = (\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, ..., \frac{\partial H}{\partial x_n})^T$$

and

$$\frac{\partial H}{\partial y} = (\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, ..., \frac{\partial H}{\partial y_n}),$$

is called a Hamiltonian system with n degrees of freedom on E.

Example

The Hamiltonian function

$$H(x, y) = x^2 - y^2 + z^2 - w^2 + 3.$$

is the energy function for the spherical pendulum

$$\begin{aligned} \dot{x} &= 2z, \\ \dot{y} &= -2w, \\ \dot{z} &= -2x, \\ \dot{w} &= -2y. \end{aligned} \tag{2}$$

Where

$$\frac{\partial H}{\partial x} = (\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, ..., \frac{\partial H}{\partial x_n})^T$$

and

$$\frac{\partial H}{\partial y} = (\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, ..., \frac{\partial H}{\partial y_n}),$$

is called a Hamiltonian system with n degrees of freedom on E.

Example

The Hamiltonian function

$$H(x, y) = x^2 - y^2 + z^2 - w^2 + 3.$$

is the energy function for the spherical pendulum

$$\begin{aligned} \dot{x} &= 2z, \\ \dot{y} &= -2w, \\ \dot{z} &= -2x, \\ \dot{w} &= -2y. \end{aligned} \quad (2)$$

Remark

All Hamiltonian systems are conservative in the sense that the Hamiltonian function or the total energy $H(x, y)$ remains constant along trajectories of the system.

Theorem (Conservation of Energy)

The total energy $H(x, y)$ of the Hamiltonian system (1) remains constant along trajectories of (1).

Remark

All Hamiltonian systems are conservative in the sense that the Hamiltonian function or the total energy $H(x, y)$ remains constant along trajectories of the system.

Theorem (Conservation of Energy)

The total energy $H(x, y)$ of the Hamiltonian system (1) remains constant along trajectories of (1).

Proof.

The total derivative of the Hamiltonian function $H(x, y)$ along a trajectory $x(t)$,$y(t)$ of (1) is

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial t} = 0.$$

Thus, $H(x, y)$ is constant along any solution curve of (1) and the trajecto- ries of (1) lie on the surfaces H(x, y) = constant. □

We next establish some very specific results about the nature of the critical points of Hamiltonian systems with one degree of freedom. Note that the equilibrium points or critical points of the system (1) correspond to the critical points of the Hamiltonian function $H(x, y)$ where $\frac{\partial H}{\partial y} = \frac{\partial H}{\partial x} = 0$. We may, without loss of generality, assume that the critical point in question has been translated to the origin.

Proof.

The total derivative of the Hamiltonian function $H(x, y)$ along a trajectory $x(t)$,$y(t)$ of (1) is

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial t} = 0.$$

Thus, $H(x, y)$ is constant along any solution curve of (1) and the trajecto- ries of (1) lie on the surfaces H(x, y) = constant. □

We next establish some very specific results about the nature of the critical points of Hamiltonian systems with one degree of freedom. Note that the equilibrium points or critical points of the system (1) correspond to the critical points of the Hamiltonian function $H(x, y)$ where $\frac{\partial H}{\partial y} = \frac{\partial H}{\partial x} = 0$. We may, without loss of generality, assume that the critical point in question has been translated to the origin.

Proof.

The total derivative of the Hamiltonian function $H(x, y)$ along a trajectory $x(t)$,$y(t)$ of (1) is

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial t} = 0.$$

Thus, $H(x, y)$ is constant along any solution curve of (1) and the trajecto- ries of (1) lie on the surfaces H(x, y) = constant. □

We next establish some very specific results about the nature of the critical points of Hamiltonian systems with one degree of freedom. Note that the equilibrium points or critical points of the system (1) correspond to the critical points of the Hamiltonian function $H(x, y)$ where $\frac{\partial H}{\partial y} = \frac{\partial H}{\partial x} = 0$. We may, without loss of generality, assume that the critical point in question has been translated to the origin.

Proof.

The total derivative of the Hamiltonian function $H(x, y)$ along a trajectory $x(t)$,$y(t)$ of (1) is

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial t} = 0.$$

Thus, $H(x, y)$ is constant along any solution curve of (1) and the trajecto- ries of (1) lie on the surfaces H(x, y) = constant. □

We next establish some very specific results about the nature of the critical points of Hamiltonian systems with one degree of freedom. Note that the equilibrium points or critical points of the system (1) correspond to the critical points of the Hamiltonian function $H(x, y)$ where $\frac{\partial H}{\partial y} = \frac{\partial H}{\partial x} = 0$. We may, without loss of generality, assume that the critical point in question has been translated to the origin.

Critical points of Hamiltonian systems

Lemma

the origin is a focus of the Hamiltonian system

$$\dot{x} = -\frac{\partial H}{\partial y}, \qquad \dot{y} = \frac{\partial H}{\partial x}. \tag{3}$$

then the origin is not a strict local maximum or minimum of the Hamilto- nian function $H(x, y)$.

Critical points of Hamiltonian systems

Lemma

the origin is a focus of the Hamiltonian system

$$\dot{x} = -\frac{\partial H}{\partial y}, \qquad \dot{y} = \frac{\partial H}{\partial x}. \tag{3}$$

then the origin is not a strict local maximum or minimum of the Hamilto- nian function $H(x, y)$.

Proof.

Suppose that the origin is a stable **focus** for (3). Then, there is an $\varepsilon > 0$ such that for $0 < r_0 < \varepsilon$ and $\theta_0 \in \mathbb{R}$, the polar coordinates of the solution of (3) with $r(0) = r_0$ and $\theta(0) = \theta_0$ satisfy $r(t, r_0, \theta_0) \longrightarrow 0$ and $|\theta(t, r_0, \theta_0)| \longrightarrow \infty$ as $t \longrightarrow +\infty$; i.e., for $(x_0, y_0) \in N_\varepsilon(0) \backsim \{0\}$, the solution $(x(t, x_0, y_0), y(t, x_0, y_0)) \longrightarrow 0$ as $t \longrightarrow +\infty$. Thus, by Theorem 1 and the continuity of $H(x, y)$ and the solution, it follows that

$$H(x_0, y_0) = lim_{t\longrightarrow+\infty} H(x(t, x_0, y_0), y(t, x_0, y_0)) = H(0, 0)$$

for all $(x_0, y_0) \in N_\varepsilon(0)$. Thus, the origin is not a strict local maximum or minimum of the function $H(x, y)$; i.e., it is not true that $H(x, y) > H(0, 0)$ or $H(x, y) < H(0, 0)$ for all points (x, y) in a deleted neighborhood of the origin. A similar argument applies when the origin is an unstable focus of (3).

Proof.

Suppose that the origin is a stable **focus** for (3). Then, there is an $\varepsilon > 0$ such that for $0 < r_0 < \varepsilon$ and $\theta_0 \in \mathbb{R}$, the polar coordinates of the solution of (3) with $r(0) = r_0$ and $\theta(0) = \theta_0$ satisfy $r(t, r_0, \theta_0) \longrightarrow 0$ and $|\theta(t, r_0, \theta_0)| \longrightarrow \infty$ as $t \longrightarrow +\infty$; i.e., for $(x_0, y_0) \in N_\varepsilon(0) \backsim \{0\}$, the solution $(x(t, x_0, y_0), y(t, x_0, y_0)) \longrightarrow 0$ as $t \longrightarrow +\infty$. Thus, by Theorem 1 and the continuity of $H(x, y)$ and the solution, it follows that

$$H(x_0, y_0) = lim_{t\longrightarrow+\infty} H(x(t, x_0, y_0), y(t, x_0, y_0)) = H(0, 0)$$

for all $(x_0, y_0) \in N_\varepsilon(0)$. Thus, the origin is not a strict local maximum or minimum of the function $H(x, y)$; i.e., it is not true that $H(x, y) > H(0, 0)$ or $H(x, y) < H(0, 0)$ for all points (x, y) in a deleted neighborhood of the origin. A similar argument applies when the origin is an unstable focus of (3).

Proof.

Suppose that the origin is a stable **focus** for (3). Then, there is an $\varepsilon > 0$ such that for $0 < r_0 < \varepsilon$ and $\theta_0 \in \mathbb{R}$, the polar coordinates of the solution of (3) with $r(0) = r_0$ and $\theta(0) = \theta_0$ satisfy $r(t, r_0, \theta_0) \longrightarrow 0$ and $|\theta(t, r_0, \theta_0)| \longrightarrow \infty$ as $t \longrightarrow +\infty$; i.e., for $(x_0, y_0) \in N_\varepsilon(0) \backsim \{0\}$, the solution $(x(t, x_0, y_0), y(t, x_0, y_0)) \longrightarrow 0$ as $t \longrightarrow +\infty$. Thus, by Theorem 1 and the continuity of $H(x, y)$ and the solution, it follows that

$$H(x_0, y_0) = lim_{t\longrightarrow+\infty} H(x(t, x_0, y_0), y(t, x_0, y_0)) = H(0, 0)$$

for all $(x_0, y_0) \in N_\varepsilon(0)$. Thus, the origin is not a strict local maximum or minimum of the function $H(x, y)$; i.e., it is not true that $H(x, y) > H(0, 0)$ or $H(x, y) < H(0, 0)$ for all points (x, y) in a deleted neighborhood of the origin. A similar argument applies when the origin is an unstable focus of (3).

□

Proof.

Suppose that the origin is a stable **focus** for (3). Then, there is an $\varepsilon > 0$ such that for $0 < r_0 < \varepsilon$ and $\theta_0 \in \mathbb{R}$, the polar coordinates of the solution of (3) with $r(0) = r_0$ and $\theta(0) = \theta_0$ satisfy $r(t, r_0, \theta_0) \longrightarrow 0$ and $|\theta(t, r_0, \theta_0)| \longrightarrow \infty$ as $t \longrightarrow +\infty$; i.e., for $(x_0, y_0) \in N_\varepsilon(0) \backsim \{0\}$, the solution $(x(t, x_0, y_0), y(t, x_0, y_0)) \longrightarrow 0$ as $t \longrightarrow +\infty$. Thus, by Theorem 1 and the continuity of $H(x, y)$ and the solution, it follows that

$$H(x_0, y_0) = lim_{t\longrightarrow+\infty} H(x(t, x_0, y_0), y(t, x_0, y_0)) = H(0, 0)$$

for all $(x_0, y_0) \in N_\varepsilon(0)$. Thus, the origin is not a strict local maximum or minimum of the function $H(x, y)$; i.e., it is not true that $H(x, y) > H(0, 0)$ or $H(x, y) < H(0, 0)$ for all points (x, y) in a deleted neighborhood of the origin. A similar argument applies when the origin is an unstable focus of (3).

□

Critical point

Definition

A critical point of the system $\dot{x} = f(x)$ at which $Df(x_0)$ has no zero eigenvalues is called a nondegenerate critical point of the system, otherwise, it is called a degenerate critical point of the system.

Racall that

Note that any nondegenerate critical point of a planar system is either a hyperbolic critical point of the system or a center of the linearized system.

Critical point

Definition

A critical point of the system $\dot{x} = f(x)$ at which $Df(x_0)$ has no zero eigenvalues is called a nondegenerate critical point of the system, otherwise, it is called a degenerate critical point of the system.

Racall that

Note that any nondegenerate critical point of a planar system is either a hyperbolic critical point of the system or a center of the linearized system.

Theorem

Any nondegenerate critical point of an analytic Hamiltonian system (3) is either a (topological) saddle or a center; furthermore, (x_0, y_0) is a (topological) saddle for (3) if it is a saddle of the Hamiltonian function $H(x, y)$ and a strict local maximum or minimum of the function $H(x, y)$ is a center for (3).

Proof.

We assume that the critical point is at the origin. Thus, $H_x(0,0) = H_y(0,0) = 0$ and the linearization of (1') at the origin is

$$\begin{pmatrix} H_{xy}(0,0) & H_{yy}(0,0) \\ -H_{xx}(0,0) & -H_{xy}(0,0) \end{pmatrix}$$

□

Theorem

Any nondegenerate critical point of an analytic Hamiltonian system (3) is either a (topological) saddle or a center; furthermore, (x_0, y_0) is a (topological) saddle for (3) if it is a saddle of the Hamiltonian function $H(x, y)$ and a strict local maximum or minimum of the function $H(x, y)$ is a center for (3).

Proof.

We assume that the critical point is at the origin. Thus, $H_x(0,0) = H_y(0,0) = 0$ and the linearization of (1') at the origin is

$$\begin{pmatrix} H_{xy}(0,0) & H_{yy}(0,0) \\ -H_{xx}(0,0) & -H_{xy}(0,0) \end{pmatrix}$$

□

We see that $trA = 0$ and that $detA = H_{xx}(0)H_{yy}(0) - H_{xy}^2(0)$. Thus, the critical point at the origin is a saddle of the function H(x, y) if det A < 0 if it is a saddle for the linear system (2) if it is a (topological) saddle for the Hamiltonian system (3).
Also, if $trA = 0$ and $detA > 0$, the origin is a center for the linear system (2). And then the origin is either a center or a focus for (3). Thus, if the nondegenerate critical point $(0, 0)$ is a strict local maximum or minimum of the function $H(x, y)$, then $detA > 0$ and, according to the above lemma, the origin is not a focus for (3); i.e., the origin is a center for the Hamiltonian system (3).

We see that $trA = 0$ and that $detA = H_{xx}(0)H_{yy}(0) - H_{xy}^2(0)$. Thus, the critical point at the origin is a saddle of the function H(x, y) if det A < 0 if it is a saddle for the linear system (2) if it is a (topological) saddle for the Hamiltonian system (3).
Also, if $trA = 0$ and $detA > 0$, the origin is a center for the linear system (2). And then the origin is either a center or a focus for (3). Thus, if the nondegenerate critical point $(0, 0)$ is a strict local maximum or minimum of the function $H(x, y)$, then $detA > 0$ and, according to the above lemma, the origin is not a focus for (3); i.e., the origin is a center for the Hamiltonian system (3).

Newtonian systems

One particular type of Hamiltonian system with one degree of freedom is the Newtonian system with one degree of freedom,

$$x'' = f(x)$$

where $f \in C^1(a, b)$. This differential equation can be written as a system in $\mathbb{R}^2$:

$$\dot{x} = y, \qquad \dot{y} = f(x). \tag{4}$$

The total energy for this system H(x, y) = T(y)+U(x) where T(y) = y2/2 is the kinetic energy and

Newtonian systems

One particular type of Hamiltonian system with one degree of freedom is the Newtonian system with one degree of freedom,

$$x'' = f(x)$$

where $f \in C^1(a, b)$. This differential equation can be written as a system in $\mathbb{R}^2$:

$$\dot{x} = y, \qquad \dot{y} = f(x). \tag{4}$$

The total energy for this system H(x, y) = T(y)+U(x) where T(y) = y2/2 is the kinetic energy and

Newtonian systems

Theorem

- ☞ *The critical points of the Newtonian system all lie on the x-axis*
- ☞ *The point $(x_0, 0)$ is a **critical point** of the Newtonian system (4) iff it is a **critical point** of the function $U(x)$, i.e., a zero of the function $f(x)$.*
- ☞ *If $(x_0, 0)$ is a strict **local maximum** of the analytic function $U(x)$, it is a saddle for (4).*

Newtonian systems

Theorem

- *The critical points of the Newtonian system all lie on the x-axis*
- *The point $(x_0, 0)$ is a* ***critical point*** *of the Newtonian system (4) iff it is a* ***critical point*** *of the function $U(x)$, i.e., a zero of the function $f(x)$.*
- *If $(x_0, 0)$ is a strict* ***local maximum*** *of the analytic function $U(x)$, it is a saddle for (4).*

Theorem

- *The critical points of the Newtonian system all lie on the x-axis*
- *The point $(x_0, 0)$ is a **critical point** of the Newtonian system (4) iff it is a **critical point** of the function $U(x)$, i.e., a zero of the function $f(x)$.*
- *If $(x_0, 0)$ is a strict **local maximum** of the analytic function $U(x)$, it is a saddle for (4).*

Newtonian systems

Theorem

- *The critical points of the Newtonian system all lie on the x-axis*
- *The point $(x_0, 0)$ is a **critical point** of the Newtonian system (4) iff it is a **critical point** of the function $U(x)$, i.e., a zero of the function $f(x)$.*
- *If $(x_0, 0)$ is a strict **local maximum** of the analytic function $U(x)$, it is a saddle for (4).*

Newtonian systems

Theorem

- ☞ *If $(x_0, 0)$ is a strict **local minimum** of the analytic function $U(x)$, it is a center for (4).*
- ☞ *If $(x_0, 0)$ is a horizontal inflection point of the function $U(x)$, it is a **cusp** for the system (4).*
- ☞ *finally, the phase portrait of (3) is symmetric with respect to the x-axis.*

Newtonian systems

Theorem

- *If $(x_0, 0)$ is a strict **local minimum** of the analytic function $U(x)$, it is a center for (4).*
- *If $(x_0, 0)$ is a horizontal inflection point of the function $U(x)$, it is a **cusp** for the system (4).*
- *finally, the phase portrait of (3) is symmetric with respect to the x-axis.*

Newtonian systems

Theorem

- *If $(x_0, 0)$ is a strict* ***local minimum*** *of the analytic function $U(x)$, it is a center for (4).*

- *If $(x_0, 0)$ is a horizontal inflection point of the function $U(x)$, it is a* ***cusp*** *for the system (4).*

- *finally, the phase portrait of (3) is symmetric with respect to the x-axis.*

Newtonian systems

Theorem

- *If $(x_0, 0)$ is a strict **local minimum** of the analytic function $U(x)$, it is a center for (4).*
- *If $(x_0, 0)$ is a horizontal inflection point of the function $U(x)$, it is a **cusp** for the system (4).*
- *finally, the phase portrait of (3) is symmetric with respect to the x-axis.*

Newtonian systems

Example

et us construct the phase portrait for the undamped pendulum

$$x'' + \sin x = 0.$$

This differential equation can be written as a Newtonian system

$$\dot{x} = y, \qquad \dot{y} = -\sin x. \tag{5}$$

where the potential energy $U(x) = \int_0^x \sin t dt = 1 - \cos x$.

Newtonian systems

Example

et us construct the phase portrait for the undamped pendulum

$$x'' + \sin x = 0.$$

This differential equation can be written as a Newtonian system

$$\dot{x} = y, \qquad \dot{y} = -\sin x. \tag{5}$$

where the potential energy $U(x) = \int_0^x \sin t dt = 1 - \cos x$.

Newtonian systems

The graph of the function $U(x)$ and the phase portrait for the undamped pendulum, which follows from Theorem 3, are shown in Figure 1 below. Note that the origin in the phase portrait for the undamped pendulum shown in Figure 1 corresponds to the stable equilibrium position of the pendulum hanging straight down. The critical points at $(\pm\pi, 0)$ correspond to the unstable equilibrium position where the pendulum is straight up. Trajectories near the origin are nearly circles and are approximated by the solution curves of the linear pendulum $x'' + x = 0$

Newtonian systems

The closed trajectories encircling the origin describe the usual periodic motions associated with a pendulum where the pendulum swings back and forth. The separatrices connecting the saddles at $(\pm\pi, 0)$ correspond to motions with total energy $H = 2$ in which case the pendulum approaches the unstable vertical position as $t \longrightarrow +\infty$.

And the trajectories outside the separatrix loops, where $H > 2$, correspond to motions where the pendulum goes over the top.

Newtonian systems

The closed trajectories encircling the origin describe the usual periodic motions associated with a pendulum where the pendulum swings back and forth. The separatrices connecting the saddles at $(\pm\pi, 0)$ correspond to motions with total energy $H = 2$ in which case the pendulum approaches the unstable vertical position as $t \longrightarrow +\infty$.
And the trajectories outside the separatrix loops, where $H > 2$, correspond to motions where the pendulum goes over the top.

Whats a Gradient system?

Definition of
Gradient system

Whats a Gradient system?

Definition of
Gradient system

Whats a Gradient system?

Definition **of**
Gradient system

Definition **of**
Gradient system

Definition

Let E be an open subset of $\mathbb{R}^n$ and let $V \in C^2(E)$. A system of the form

$$\dot{x} = -\nabla V(x). \qquad (6)$$

where $\nabla V(x) = (\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots \frac{\partial V}{\partial x_n})^T$, is called a gradient system on E.

Definition

Let E be an open subset of $\mathbb{R}^n$ and let $V \in C^2(E)$. A system of the form

$$\dot{x} = -\nabla V(x). \tag{6}$$

where $\nabla V(x) = (\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots \frac{\partial V}{\partial x_n})^T$, is called a gradient system on E.

Note that the equilibrium points or critical points of the gradient system (4) correspond to the critical points of the function $V(x)$ where $gradV(x) = 0$.

Points where grad $V(x) \neq 0$ are called regular points of the function $V(x)$.

At regular points of $V(x)$, the gradient vector $gradV(x)$ is perpendicular to the level surface $V(x) = constant$ through the point.And it is easy to show that at a critical point x_0 of $V(x)$, which is a strict local minimum of $V(x)$, the function $V(x) - V(x_0)$ is a strict Liapunov function for the system (4) in some neighborhood of x_0; We therefore have the following theorem:

Note that the equilibrium points or critical points of the gradient system (4) correspond to the critical points of the function $V(x)$ where $gradV(x) = 0$.
Points where grad $V(x) \neq 0$ are called regular points of the function $V(x)$.
At regular points of $V(x)$, the gradient vector $gradV(x)$ is perpendicular to the level surface $V(x) = constant$ through the point.And it is easy to show that at a critical point x_0 of $V(x)$, which is a strict local minimum of $V(x)$, the function $V(x) - V(x_0)$ is a strict Liapunov function for the system (4) in some neighborhood of x_0; We therefore have the following theorem:

Note that the equilibrium points or critical points of the gradient system (4) correspond to the critical points of the function $V(x)$ where $gradV(x) = 0$.
Points where grad $V(x) \neq 0$ are called regular points of the function $V(x)$.
At regular points of $V(x)$, the gradient vector $gradV(x)$ is perpendicular to the level surface $V(x) = constant$ through the point. And it is easy to show that at a critical point x_0 of $V(x)$, which is a strict local minimum of $V(x)$, the function $V(x) - V(x_0)$ is a strict Liapunov function for the system (4) in some neighborhood of x_0; We therefore have the following theorem:

Note that the equilibrium points or critical points of the gradient system (4) correspond to the critical points of the function $V(x)$ where $gradV(x) = 0$.
Points where grad $V(x) \neq 0$ are called regular points of the function $V(x)$.
At regular points of $V(x)$, the gradient vector $gradV(x)$ is perpendicular to the level surface $V(x) = constant$ through the point.And it is easy to show that at a critical point x_0 of $V(x)$, which is a strict local minimum of $V(x)$, the function $V(x) - V(x_0)$ is a strict Liapunov function for the system (4) in some neighborhood of x_0; We therefore have the following theorem:

Note that the equilibrium points or critical points of the gradient system (4) correspond to the critical points of the function $V(x)$ where $gradV(x) = 0$.
Points where grad $V(x) \neq 0$ are called regular points of the function $V(x)$.
At regular points of $V(x)$, the gradient vector $gradV(x)$ is perpendicular to the level surface $V(x) = constant$ through the point.And it is easy to show that at a critical point x_0 of $V(x)$, which is a strict local minimum of $V(x)$, the function $V(x) - V(x_0)$ is a strict Liapunov function for the system (4) in some neighborhood of x_0; We therefore have the following theorem:

Theorem

Any nondegenerate critical point of an analytic gradient system (4) on $\mathbb{R}^2$ is either a saddle or a node; furthermore, if (x_0, y_0) is a saddle of the function $V(x, y)$, it is a saddle of (4) and if (x_0, y_0) is a strict local maximum or minimum of the function $V(x, y)$, it is respectively an unstable or a stable node for (4).

Theorem

Any nondegenerate critical point of an analytic gradient system (4) on $\mathbb{R}^2$ is either a saddle or a node;
furthermore, if (x_0, y_0) is a saddle of the function $V(x, y)$, it is a saddle of (4) and if (x_0, y_0) is a strict local maximum or minimum of the function $V(x, y)$, it is respectively an unstable or a stable node for (4).

Theorem

Any nondegenerate critical point of an analytic gradient system (4) on $\mathbb{R}^2$ is either a saddle or a node;
furthermore, if (x_0, y_0) is a saddle of the function $V(x, y)$, it is a saddle of (4) and if (x_0, y_0) is a strict local maximum or minimum of the function $V(x, y)$, it is respectively an unstable or a stable node for (4).

Example (*)

et $V(x, y) = x^2(x-1)^2 + y^2$. The gradient system (4) then has the form

$$\dot{x} = -4x(x-1)(x-1/2), \qquad \dot{y} = -2y. \tag{7}$$

There are critical points at $(0, 0)$, $(1/2, 0)$ and $(1, 0)$. It follows from the Theorem that $(0, 0)$ and $(1, 0)$ are stable nodes and that $(1/2, 0)$ is a saddle for this system;. The level curves $V(x, y) = constant$ and the trajectories of this system are shown in Figure 2.

Example (*)

et $V(x, y) = x^2(x-1)^2 + y^2$. The gradient system (4) then has the form

$$\dot{x} = -4x(x-1)(x-1/2), \qquad \dot{y} = -2y. \tag{7}$$

There are critical points at $(0, 0)$, $(1/2, 0)$ and $(1, 0)$. It follows from the Theorem that $(0, 0)$ and $(1, 0)$ are stable nodes and that $(1/2, 0)$ is a saddle for this system;. The level curves $V(x, y) = \textit{constant}$ and the trajectories of this system are shown in Figure 2.

Example (*)

et $V(x,y) = x^2(x-1)^2 + y^2$. The gradient system (4) then has the form

$$\dot{x} = -4x(x-1)(x-1/2), \qquad \dot{y} = -2y. \tag{7}$$

There are critical points at $(0,0)$, $(1/2,0)$ and $(1,0)$. It follows from the Theorem that $(0,0)$ and $(1,0)$ are stable nodes and that $(1/2,0)$ is a saddle for this system;. The level curves $V(x,y) = constant$ and the trajectories of this system are shown in Figure 2.

Example (*)

et $V(x,y) = x^2(x-1)^2 + y^2$. The gradient system (4) then has the form

$$\dot{x} = -4x(x-1)(x-1/2), \qquad \dot{y} = -2y. \tag{7}$$

There are critical points at $(0,0)$, $(1/2,0)$ and $(1,0)$. It follows from the Theorem that $(0,0)$ and $(1,0)$ are stable nodes and that $(1/2,0)$ is a saddle for this system;. The level curves $V(x,y) = constant$ and the trajectories of this system are shown in Figure 2.

There is an interesting relationship between gradient and Hamiltonian systems, is considered in this section.

Definition

Consider the planar system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y). \tag{8}$$

The **system orthogonal** to (**??**) is defined as the syste

$$\dot{x} = -Q(x, y), \qquad \dot{y} = P(x, y). \tag{9}$$

It is Clear that, (**??**) and (**??**) have the same **critical points** and at regular points, trajectories of (**??**) are orthogonal to the trajectories of (**??**). Furthermore, centers of (**??**) correspond to nodes of (**??**), saddles of (**??**) correspond to saddles of (**??**), and foci of (**??**) correspond to foci of (**??**).
Also, if (**??**) is a Hamiltonian system with $P = H_y$, and $Q = -H_x$, then (**??**) is a gradient system and conversely.

Theorem

*The system (**??**) is a Hamiltonian system if the system (**??**) orthogonal to (**??**) is a gradient system.*
In higher dimensions, we have that if (1) (Hami syst) is a Hamiltonian system with n degrees of freedom then the system

$$\dot{x} = -\frac{\partial H}{\partial y}, \qquad \dot{y} = \frac{\partial H}{\partial x}. \tag{10}$$

*orthogonal to (1) is a gradient system in $\mathbb{R}^{2n}$ and the trajectories of the gradient system (**??**) cross the surfaces $H(x, y) =$ constant orthogonally.*
In Example (*) if we take $H(x, y) = V(x, y)$, then Figure 2 illustrates the orthogonality of the trajectories of the Hamiltonian and gradient flows, the Hamiltonian flow swirling clockwise.

The proof (Exercice)